Sub-leading Soft Photons and Gravitons

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Outline

- What are soft bosons? "Soft" $=$ "low energy"
  - Soft Theorems (incl. sub-leading terms) – particle picture
  - Asymptotic space-time structure – field picture
  - Would like to unify these pictures – sub-leading terms important!
- Recent *classical* understanding – soft theorems come from boundary terms (CD, J. Wilson-Gerow, PCE Stamp – arXiv:2012.13356 [hep-th], to appear in JHEP.)
- How is classical intuition reflected in quantum mechanical computations?
- What comes next? (If there is time I will speculate)

Soft Bosons

Soft particles are just particles with energy approaching zero ($\lambda \to \infty$).

Depending on application, can mean strict $\lim_{|\mathbf{q}| \to 0}$, or simply that $\lambda \gg \{L\}$, or $|\mathbf{q}| \ll \{E\}$.

The Soft Photon Theorem (see e.g. Weinberg, 1965)

N particles with momenta $\{p_n\}$ scatter into a set of M particles with momenta $\{p'_m\}$.

The amplitude is $A(\{p'_m\}|\{p_n\})$.

Now consider the amplitude for the same process, but add to the out state a photon with momentum q and polarization vector ϵ. One finds that, the amplitude becomes approximately:

$$\left(\mathcal{S}_{(0)} + \mathcal{S}_{(1)}\right) A(\{p'_m\}|\{p_n\}),$$

as $|\mathbf{q}| \to 0$. This is soft factorization!
Leading order is exact at tree level, $\mathcal{S}_{(1)}$ gets loop corrections.

The Soft Photon Theorem

More concretely, we have

$$\mathcal{S}_{(0,1)} = \mathcal{N}_a \left[\sum_{m=1}^{M} e_m \mathcal{S}^a_{(0,1)}(q, \hat{x}'_m, p'_m) - \sum_{n=1}^{N} e_n \mathcal{S}^a_{(0,1)}(q, \hat{x}_n, p_n) \right],$$

where $\mathcal{N}_a \propto \bar{\epsilon}_a$ is the "wavefunction" of the outgoing photon, the $e_{m,n}$ are the charges of the matter particles, and we have defined

$$\mathcal{S}^a_{(0)}(q,p) \equiv \frac{p^a}{q \cdot p} \sim \mathcal{O}(|\mathbf{q}|^{-1})$$
$$\mathcal{S}^a_{(1)}(q,\hat{x},p) \equiv i\frac{q_b \hat{J}^{ba}}{q \cdot p} \sim \mathcal{O}(1).$$

Here $\hat{J}^{ab} \equiv p^a \hat{x}^b - \hat{x}^a p^b \equiv 2p^{[a}\hat{x}^{b]}$ is the angular momentum operator.

The Soft Graviton Theorem

Instead, add to the out state a graviton with momentum q and polarization tensor ϵ. At tree level, the amplitude becomes:

$$\left(\mathscr{S}_{(0)} + \mathscr{S}_{(1)} + \mathscr{S}_{(2)}\right) A(\{p'_m\}|\{p_n\})$$

near $|\mathbf{q}| = 0$.

Note that in the gravitational case there are three "soft factors" (e.g. Cachazo & Strominger arXiv:1404.4091).

The Soft Graviton Theorem

Once more,

$$\mathscr{S}_{(0,1,2)} = \kappa \mathscr{N}_{ab} \left[\sum_{m=1}^{M} \mathscr{S}^{ab}_{(0,1,2)}(q, \hat{x}'_m, p'_m) - \sum_{n=1}^{N} \mathscr{S}^{ab}_{(0,1,2)}(q, \hat{x}_n, p_n) \right],$$

in which $\kappa \equiv \sqrt{8\pi G}$ is the Newton coupling, $\mathscr{N}_{ab} \propto \bar{\epsilon}_{ab}$ the outgoing graviton "wavefunction," and

$$\mathscr{S}^{ab}_{(0)}(q, p) \equiv \frac{p^a p^b}{q \cdot p}$$

$$\mathscr{S}^{ab}_{(1)}(q, \hat{x}, p) \equiv i \frac{q_c \hat{J}^{c(a} p^{b)}}{q \cdot p}$$

$$\mathscr{S}^{ab}_{(2)}(q, \hat{x}, p) \equiv -\frac{1}{2} \frac{q_c \hat{J}^{ac} q_d \hat{J}^{bd}}{q \cdot p},$$

where the parentheses around indices indicate symmetrization.

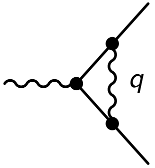
The Soft Theorems

Questions:

- What to do with IR divergence in leading order terms, which are $\propto 1/|\mathbf{q}|$?
- **Why are there 3 soft factors for gravity, but only 2 in QED?**

IR divergences in QFT

Virtual bosons in amplitudes can be soft too:



IR divergence in vertex when $|\mathbf{q}| \to 0$, exponentiates and sets probability for process to zero (Bloch & Nordsieck, Weinberg)

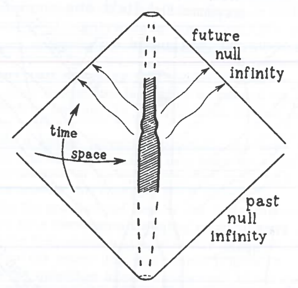
Two solutions:

- Inclusive approach – also sum up diagrams with *real* soft radiation
- Dressed state approach – treat soft radiation as part of IN/OUT states

IR divergences cancel, but both approaches involve an infinite number of real soft bosons – non-perturbative? Sub-leading effects?

Asymptotic Symmetries

Soft theorems related to gauge (diffeo) symmetry at null infinity.



Sketch taken from R. Penrose Collected Works

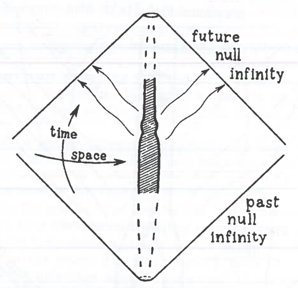
Gauge: $A_a \to A_a + \nabla_a \Lambda$
Diffeo: $x^a \to x^a + \xi^a$
At null infinity,

- $\Lambda(x) \to \lambda(\theta, \phi)$
- $\xi : u \to u + \alpha(\theta, \phi)$

u is time coordinate along null infinity. α called "supertranslation"

Asymptotic Symmetries

Soft theorems related to gauge (diffeo) symmetry at null infinity.



Sketch taken from R. Penrose Collected Works

Charges Q_λ, Q_α generate transformations $\lambda(\theta, \phi)$, $\alpha(\theta, \phi)$.

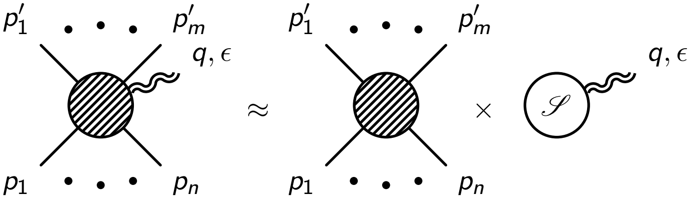
Conservation of Q_λ, Q_α between future and past null infinity equivalent to leading order soft theorems!

Dressed states are those with $\hat{Q}|\Psi\rangle = 0$.

What about sub-leading terms? Lots of work on e.g. "superrotations," but still problematic.

Picturing the Soft Theorems

For small $|\mathbf{q}|$,



How can we get factorization? When $\mathscr{S}$ depends only on data at endpoints of worldlines!

Picturing the Soft Theorems

At the level of the action, matter couples to the fields via

$$S_{int} = \int d^4x\, A_a(x) j^a(x) = \int \frac{d^4q}{(2\pi)^4} A_a(q) j^a(-q)$$

OR

$$S_{int} = -\frac{1}{2}\int d^4x\, h_{ab}(x) T^{ab}(x) = -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4} h_{ab}(q) T^{ab}(-q)$$

A_a or h_{ab} can be thought of as representing a real "soft boson" if we take the momentum q^a to be on-shell ($q \cdot q = 0$), and of very low frequency. The matter then couples to the soft fields via

$$j^a(-q), \quad T^{ab}(-q)$$

with $|\mathbf{q}|$ small.

Electromagnetism

The conserved current for a particle with charge e coupled to the Maxwell field, which follows a classical trajectory $X^a(s)$, is

$$j^a(x) \; = \; e\int_0^\infty ds\, \dot{X}^a(s)\, \delta^{(4)}(x - X(s)),$$

s: the particle's proper time, and $\dot{X}^a(s) \equiv \frac{d}{ds}X^a(s)$.

In momentum space,

$$j^a(q) \; = \; e\int_0^\infty ds\, \dot{X}^a(s)\, e^{iq\cdot X(s)}.$$

Electromagnetism

$$j^a(q) \;=\; e\int_0^\infty ds\,\dot{X}^a(s)\,e^{iq\cdot X(s)}$$

$q_a j^a(q) \neq 0!$ Let's rewrite this:

$$j^a(q) \;=\; e\int ds\,\dot{X}^a\left(\frac{1}{iq\cdot\dot{X}}\right)\frac{d}{ds}e^{iq\cdot X}$$

Integrating by parts in s gives

$$j^a(q) \;=\; -ie\int ds\,\frac{d}{ds}\left(e^{iq\cdot X}\frac{\dot{X}^a}{q\cdot\dot{X}}\right) + ie\int ds\,e^{iq\cdot X}\frac{d}{ds}\left(\frac{\dot{X}^a}{q\cdot\dot{X}}\right).$$

Drop the first term!

Electromagnetism: Leading Order

Taylor expansion of phase,

$$j^a(q) \;=\; ie\int ds\left[\sum_{k=0}^{\infty}\frac{1}{k!}(iq\cdot X)^k\right]\frac{d}{ds}\left(\frac{\dot{X}^a}{q\cdot\dot{X}}\right)\equiv\sum_{k=0}^{\infty}j^a_{(k)}(q)$$

And we just look at the first term ($k=0$):

$$j^a_{(0)}(q) \;=\; ie\int ds\,\frac{d}{ds}\left(\frac{\dot{X}^a}{q\cdot\dot{X}}\right)=ie\Delta\left(\frac{\dot{X}^a}{q\cdot\dot{X}}\right),$$

with $\Delta f(s)\equiv f(s\to\infty)-f(s\to 0)$.

Electromagnetism: Leading Order

$$j^a_{(0)}(q) \;=\; ie \int ds \, \frac{d}{ds} \left(\frac{\dot{X}^a}{q \cdot \dot{X}} \right) = ie\Delta \left(\frac{\dot{X}^a}{q \cdot \dot{X}} \right)$$

Leading soft factor:

$$\mathcal{S}^a_{(0)}(q, p) \equiv \frac{p^a}{q \cdot p}$$

Current bdry. term is the soft factor:

$$i\, j^a_{(0)}(-q) \;=\; e\Delta \mathcal{S}^a_{(0)}(q, m\dot{X}).$$

The Δ here even explains the relative minus sign between outgoing and incoming particles in the soft theorem!

Electromagnetism: Sub-leading Order

$$j^a_{(1)}(q) \;=\; ie \int ds\,(iq\cdot X)\frac{d}{ds}\left(\frac{\dot{X}^a}{q\cdot\dot{X}}\right)$$

Sub-leading soft factor:

$$\mathcal{S}^a_{(1)}(q,\hat{x},p) \equiv i\,\frac{q_b \hat{J}^{ba}}{q\cdot p}$$

After more integration by parts...

Current bdry. term is the soft factor:

$$i\,j^a_{(1)}(-q) \;=\; e\Delta\mathcal{S}^a_{(1)}(q,X,m\dot{X}).$$

Electromagnetism: (Sub)^k-leading Order?

$$j^a_{(k)}(q) \;=\; \frac{ie}{k!}\int ds\,(iq\cdot X)^k\,\frac{d}{ds}\left(\frac{\dot{X}^a}{q\cdot\dot{X}}\right)$$

$$= \;\frac{(i^{k+1})e}{k!}\int ds\left[\frac{d}{ds}\left((q\cdot X)^k\,\frac{\dot{X}^a}{q\cdot\dot{X}} - k(q\cdot X)^{k-1}X^a\right)\right.$$
$$\left. + \,k(k-1)(q\cdot X)^{k-2}(q\cdot\dot{X})X^a\right]$$

"Factorization error" $\propto k(k-1)$, vanishes at leading and sub-leading order *only*! $\rightarrow$ **No further soft photon theorems**!

Linearized Gravity

The gravitational current for a particle with charge mass m following trajectory $X^a(s)$ is the stress tensor

$$T^{ab}(x) \; = \; m \int ds \, \dot{X}^a(s) \dot{X}^b(s) \, \delta^{(4)}(x - X(s))$$

and, as in the electromagnetic case, we are interested in the Fourier transform of this:

$$T^{ab}(q) \; = \; m \int ds \, \dot{X}^a(s) \dot{X}^b(s) \, e^{iq \cdot X(s)}$$

Linearized Gravity

Just as before, int. by parts and drop problematic bdry. term, giving

$$T^{ab}(q) \; = \; im \int ds \, e^{iq\cdot X} \frac{d}{ds} \left(\frac{\dot{X}^a \dot{X}^b}{q \cdot \dot{X}} \right)$$

Expand as

$$T^{ab}(q) \; = \; im \int ds \left[\sum_{k=0}^{\infty} \frac{1}{k!} (iq \cdot X)^k \right] \frac{d}{ds} \left(\frac{\dot{X}^a \dot{X}^b}{q \cdot \dot{X}} \right) \; \equiv \; \sum_{k=0}^{\infty} T^{ab}_{(k)}(q),$$

Linearized Gravity: Leading Order

The leading order term is again straightforward

$$T^{ab}_{(0)}(q) \;=\; im \int ds \, \frac{d}{ds} \left(\frac{\dot{X}^a \dot{X}^b}{q \cdot \dot{X}} \right) \;=\; im\Delta \left(\frac{\dot{X}^a \dot{X}^b}{q \cdot \dot{X}} \right)$$

Recall

$$\mathscr{S}^{ab}_{(0)}(q, p) \equiv \frac{p^a p^b}{q \cdot p}$$

Stress tensor bdry. term is the soft factor:

$$iT^{ab}_{(0)}(-q) = \Delta \mathscr{S}^{ab}_{(0)}(q, m\dot{X})$$

Leading order works just as in QED.

Linearized Gravity: Sub-Leading Order

$$T^{ab}_{(1)}(q) \;=\; im \int ds\,(iq\cdot X)\frac{d}{ds}\left(\frac{\dot{X}^a\dot{X}^b}{q\cdot\dot{X}}\right)$$

With $P^a = m\dot{X}^a$,

$$= \Delta\left(\frac{q_c J^{c(a}P^{b)}}{q\cdot P}\right) - m\int ds\, X^{(a}\ddot{X}^{b)}$$

Recall

$$\mathscr{S}^{ab}_{(1)}(q,\hat{x},p) \equiv i\,\frac{q_c\hat{J}^{c(a}p^{b)}}{q\cdot p}$$

Stress tensor bdry. term is NOT the soft factor:

$$iT^{ab}_{(1)}(-q) \neq \Delta\mathscr{S}^{ab}_{(1)}(q,X,m\dot{X})$$

...Crap.

Linearized Gravity: Doesn't Work?

What to make of $-m \int ds\, X^{(a}\ddot{X}^{b)}$?

Look at stress-energy conservation.

$T^{ab}_{,a} = 0$ implies that this term vanishes on-shell (T not conserved identically).

$$T^{ab}{}_{,a} \;=\; -m \int ds\, \ddot{X}^b \delta^{(4)}(x - X(s)) = 0$$

$$T^{ab}_{,a}(q) \;=\; -m \int ds\, e^{iq\cdot X} \ddot{X}^b = 0$$

Expand phase again. At sub-leading order get (q^a arbitrary)

$$-imq_a \int ds\, X^a \ddot{X}^b = 0$$

Remember this is **linearized** gravity!

Linearized Gravity: Sub-Leading Order

$$T^{ab}_{(1)}(q) \;=\; im \int ds\,(iq\cdot X)\frac{d}{ds}\left(\frac{\dot{X}^a\dot{X}^b}{q\cdot\dot{X}}\right)$$

With $P^a = m\dot{X}^a$,

$$= \Delta\left(\frac{q_c J^{c(a}P^{b)}}{q\cdot P}\right) - m\int ds\, X^{(a}\ddot{X}^{b)}$$

Recall

$$\mathscr{S}^{ab}_{(1)}(q,\hat{x},p) \equiv i\,\frac{q_c \hat{J}^{c(a}p^{b)}}{q\cdot p}$$

Stress tensor bdry. term IS ACTUALLY the soft factor:

$$iT^{ab}_{(1)}(-q) = \Delta\mathscr{S}^{ab}_{(1)}(q,X,m\dot{X})$$

... plus a term which vanishes on-shell.

Linearized Gravity: Sub-Sub-Leading Order

$$T^{ab}_{(2)}(q) \;=\; \frac{im}{2}\int ds\,(iq\cdot X)^2\frac{d}{ds}\left(\frac{\dot{X}^a\dot{X}^b}{q\cdot\dot{X}}\right)$$

With $P^a = m\dot{X}^a$,

$$= -\frac{i}{2}\Delta\left(\frac{q_c J^{ac} q_d J^{bd}}{q\cdot P}\right) + \frac{im}{2}\int ds\,\left[X^a X^b(q\cdot\ddot{X}) + 2(q\cdot X)X^{(a}\ddot{X}^{b)}\right].$$

Recall

$$\mathscr{S}^{ab}_{(2)}(q,\hat{x},p) \equiv -\frac{1}{2}\frac{q_c\hat{J}^{ac}q_d\hat{J}^{bd}}{q\cdot p}$$

Stress tensor bdry. term is the soft factor:

$$iT^{ab}_{(2)}(-q) = \Delta\mathscr{S}^{ab}_{(2)}(q,X,m\dot{X})$$

... plus a term which vanishes on-shell.

Linearized Gravity: (Sub)^k-leading Order?

$$T^{ab}_{(k)}(q) \;=\; \frac{im}{k!}\int ds\,\left[(iq\cdot X)^k\right]\frac{d}{ds}\left(\frac{\dot{X}^a\dot{X}^b}{q\cdot\dot{X}}\right)$$

$$= \;\frac{(i^{k+1})m}{k!}\int ds\left[\frac{d}{ds}\left((q\cdot X)^k\frac{\dot{X}^a\dot{X}^b}{q\cdot\dot{X}}\right.\right.$$

$$\left.-\frac{k}{2}(q\cdot X)^{k-1}\frac{d}{ds}(X^aX^b)+\frac{k}{2}X^aX^b\frac{d}{ds}(q\cdot X)^{k-1}\right)$$

$$+\,k(q\cdot X)^{k-1}X^{(a}\ddot{X}^{b)}-\frac{1}{2}k(k-1)X^aX^b(q\cdot X)^{k-2}(q\cdot\ddot{X})$$

$$\left.-\,\frac{1}{2}k(k-1)(k-2)X^aX^b(q\cdot\dot{X})^2(q\cdot X)^{k-3}\right]$$

"Factorization error" $\propto k(k-1)(k-2)$, nonzero after sub-sub-leading order! $\rightarrow$ **No further soft graviton theorems**!

The Soft Theorems

Questions:

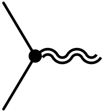
- What to do with IR divergence in leading order terms, which are $\propto 1/|\mathbf{q}|$?
  - These are actually helpful. Used to cancel virtual divergences.
- **Why are there 3 soft factors for gravity, but only 2 in QED?**
  - **Because there are exactly that many boundary terms in the respective particle currents, and these dominate at low $|\mathbf{q}|$.**
- How does these being boundary terms imply that they factorize quantum mechanically?

Factorization at the Quantum Level

Look at correlator in graviton background: $\langle \phi(x')\phi(x) \rangle_{h_{ab}}$

Path integral representation:

$$i \int_0^\infty ds \int_x^{x'} \mathcal{D}X(s')\, e^{iS[X]|_0^s + i\kappa \int T^{ab} h_{ab}|_0^s}$$



$$\sim i \int_0^\infty ds \int_x^{x'} \mathcal{D}X(s')\, e^{iS[X]|_0^s} \left[i\kappa \int T^{ab} h_{ab}|_0^s \right]$$

Up to sub-sub-leading order, $T^{ab}(x, x', \partial_x, \partial_{x'}) \to$ can pull out of path integral! After LSZ, factors out of tree amplitudes.

Next Steps?

- Use this framework to derive/verify loop corrections?
- Go beyond linearized gravity?
- Sub-leading terms are associated with boundary, **are they explained by asymptotic symmetries?**
  - No IR divergences forcing us to use e.g. sub-leading dressed states.

Thank you!

And remember: always keep boundary terms when you integrate by parts!